Potentials Which Cause the Same Scattering at all Energies in One Dimension

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Explicit scattering solutions of the one-dimensional Schrödinger equation are given. A one-parameter family of the potentials considered here causes the same scattering at all energies. The previously published explicit examples of nonuniqueness in the one-dimensional inverse quantum problem are special cases of the potentials given here.

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Consider the one-dimensional Schrödinger equation,

\[ d^2\psi(k,x)/dx^2 + k^2\psi(k,x) = V(x)\psi(k,x). \]

If the potential \( V(x) \) vanishes as \( x \to \pm \infty \) in some sense, we find two linearly independent solutions \( \psi_l \) and \( \psi_r \), which are usually called physical solutions from the left and from the right respectively, with the boundary conditions

\[
\begin{pmatrix}
\psi_l(k,x) \\
\psi_r(k,x)
\end{pmatrix} =
\begin{bmatrix}
T(k) \\
R(k)
\end{bmatrix} e^{ikx} +
\begin{bmatrix}
0 \\
1
\end{bmatrix} e^{-ikx} + o(1), \text{ as } x \to \infty,
\]

and

\[
\begin{pmatrix}
\psi_l(k,x) \\
\psi_r(k,x)
\end{pmatrix} =
\begin{bmatrix}
L(k) \\
T(k)
\end{bmatrix} e^{-ikx} +
\begin{bmatrix}
1 \\
0
\end{bmatrix} e^{ikx} + o(1), \text{ as } x \to -\infty,
\]

where

\[
S(k) =
\begin{bmatrix}
T(k) & R(k) \\
L(k) & T(k)
\end{bmatrix}
\]

is the scattering matrix, \( T(k) \) is the transmission coefficient, and \( R(k) \) and \( L(k) \) are the reflection coefficients from the right and from the left, respectively. Good reviews of the scattering and inverse scattering problem for the Schrödinger equation exist in the literature.\[1\text{-}3\]

Letting

\[ m_l(k,x) = \frac{1}{T(k)} e^{-ikx} \psi_l(k,x) \]

and

\[ m_r(k,x) = \frac{1}{T(k)} e^{ikx} \psi_r(k,x) \]

we obtain\[4\]

\[ d^2m_l(k,x)/dx^2 + 2ik dm_l(k,x)/dx = V(x)m_l(k,x) \]

and

\[ d^2m_r(k,x)/dx^2 - 2ik dm_r(k,x)/dx = V(x)m_r(k,x) , \]

with the boundary conditions

\[ m_l(k,x) = 1 + o(1) \text{ and } dm_l(k,x)/dx = o(1), \text{ as } x \to \infty, \]

\[ m_r(k,x) = 1 + o(1) \text{ and } dm_r(k,x)/dx = o(1), \text{ as } x \to -\infty. \]

If we let

\[ m_l(k,x) = \sum_{n=0}^{\infty} \left( \frac{i}{k} \right)^n f_n(x) \text{ and } m_r(k,x) = \sum_{n=0}^{\infty} \left( -\frac{i}{k} \right)^n g_n(x), \]

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we obtain
\[ f_0(x) = 1; \quad f_n(x) = \frac{1}{2} \frac{d}{dx} f_{n-1}(x) + \frac{1}{2} \int_x^\infty dy \, V(y) f_{n-1}(y), \quad n \geq 1; \]
\[ (1) \]
and
\[ g_0(x) = 1; \quad g_n(x) = \frac{1}{2} \frac{d}{dx} g_{n-1}(x) - \frac{1}{2} \int_x^- dy \, V(y) g_{n-1}(y), \quad n \geq 1. \]
\[ (2) \]
Consider the family of potentials \( V(x, a, b, c, M, N) \) defined as
\[ V(x, a, b, c, M, N) = c \delta(x) - 2 \theta(x) [P(x, a, N)/P(x, a, N)'] - 2 \theta(-x) [Q'(x, \beta, M)/Q(x, \beta, M)'], \]
where \( a, b, \) and \( c \) are real parameters, \( M \) and \( N \) are positive integers, \( \delta(x) \) is the Dirac delta function, \( \theta(x) \) is the Heaviside step function, the prime denotes the \( x \) derivative, and
\[ P(x, a, N) \equiv (x+1)^{N(N+1)/2} + a(x+1)^{(N-2)(N-1)/2}, \]
\[ (3) \]
and
\[ Q(x, \beta, M) \equiv (-x+1)^{M(M+1)/2} + \beta(-x+1)^{(M-2)(M-1)/2}. \]
\[ (4) \]
The choice of 1 in \((\pm x \pm 1)\) in (3) and (4) is arbitrary, but this choice causes no loss of generality.

From (1) and (2) we obtain
\[ \theta(x)m_l(k, x, a, N) = \sum_{n=0}^N \left( \frac{i}{k} \right)^n \left[ \frac{(N+n)! (x+1)^{N(N+1)/2-n}}{2^n! (N-n)!} + a \theta(N - \frac{x}{2} - n) \frac{(N+n-2)! (x+1)^{(N-2)(N-1)/2-n}}{2^n! (N-n-2)!} \right] \frac{1}{P(x, a, N)}, \]
\[ (5) \]
and
\[ \theta(-x)m_r(k, x, a, N) = \sum_{n=0}^N \left( \frac{i}{k} \right)^n \left[ \frac{(M+n)! (-x+1)^{M(M+1)/2-n}}{2^n! (M-n)!} + \beta \theta(M - \frac{x}{2} - n) \frac{(M+n-2)! (-x+1)^{(M-2)(M-1)/2-n}}{2^n! (M-n-2)!} \right] \frac{1}{Q(x, \beta, M)}. \]
\[ (6) \]
Using (5) and (6), we can write the physical solutions as
\[ \psi_l(k, x, a, b, c, M, N) = \theta(x) T(k) e^{ikx} m_l(k, x, a, N) + \theta(-x) [e^{ikx} m_r(-k, x, b, M) + L(k) e^{-ikx} m_r(k, x, b, M)], \]
and
\[ \psi_r(k, x, a, b, c, M, N) = \theta(x) [e^{-ikx} m_l(-k, x, a, N) + R(k) e^{ikx} m_l(k, x, a, N)] + \theta(-x) T(k) e^{-ikx} m_r(k, x, b, M), \]
where the transmission and reflection coefficients are to be determined from the boundary conditions
\[ \lim_{x \to 0^+} - \lim_{x \to 0^-} \begin{bmatrix} \psi_l(k, x, a, b, c, M, N) \\ \psi_r(k, x, a, b, c, M, N) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \]
and
\[ \lim_{x \to 0^+} - \lim_{x \to 0^-} \frac{d}{dx} \begin{bmatrix} \psi_l(k, x, a, b, c, M, N) \\ \psi_r(k, x, a, b, c, M, N) \end{bmatrix} = e \lim_{x \to 0} \begin{bmatrix} \psi_l(k, x, a, b, c, M, N) \\ \psi_r(k, x, a, b, c, M, N) \end{bmatrix}. \]

Hence we obtain
\[ T(k) = 2ik/D(k, a, b, c, M, N), \]
and
\[ L(k) = E(k, a, b, c, M, N)/D(k, a, b, c, M, N), \]
\[ 2160 \]
and

\[ R(k) = E(-k, a, \beta, c, M, N)/D(k, a, \beta, C, M, N), \]

where we have defined

\[ D(k, a, \beta, c, M, N) \equiv (2ik - c)m_I(k, 0, a, N)m_r(k, 0, \beta, M) + m_r(k, 0, \beta, M)dm_I(k, 0, a, N)/dx \]
\[ - m_I(k, 0, a, N)dm_r(k, 0, \beta, M)/dx, \]

(7)

and

\[ E(k, a, \beta, c, M, N) \equiv cm_I(k, 0, a, N)m_r(-k, 0, \beta, M) + m_r(k, 0, a, N)dm_r(-k, 0, \beta, M)/dx \]
\[ - m_I(-k, 0, \beta, M)dm_I(k, 0, a, N)/dx. \]

(8)

If we let

\[ c + [M - (M - 1)\beta]/(1 + \beta) + [N - (N - 1)a]/(1 + a) = 0, \]

(9)

both \( D(k, a, \beta, c, M, N) \) and \( E(k, a, \beta, c, M, N) \) become independent of \( a \) and \( \beta \); this can be seen by use of (5), (6), and (9) and by differentiation of (7) and (8) with respect to one of the parameters \( a \) and \( \beta \). Thus, although the family of potentials \( V(x, a, \beta, c, M, N) \) still contains one of the parameters \( a \) and \( \beta \) as an arbitrary parameter, the corresponding scattering matrix becomes independent of both \( a \) and \( \beta \).

The previously published nonuniqueness examples in the one-dimensional inverse quantum scattering are all special cases of the family \( V(x, a, \beta, c, N, M) \) considered here: \( c = -2, M = 1, N = 1 \); \( c = -1, M = 3, N = 3 \); \( c = 0, N = 1, M = 2 \); \( c = 1, N = 1, M = 3 \).

As a special case, \( c+1/(1+\beta)+(N-(N-1)a)/(1+a)=0 \), we obtain

\[ D(k, a, \beta, c, N, 1) = 2ik + \sum_{n=0}^{N-1} \left( \frac{i}{k} \right)^n \frac{c - N(N-1)}{n+1} \frac{(N+n-1)!}{2^n!N!(N-n-1)!} \]

and

\[ E(k, a, \beta, c, N, 1) = \sum_{n=0}^{N-1} \left( \frac{i}{k} \right)^n \frac{(c-n)(N+n-1)!}{2^n!(N-n-1)!}. \]

The ambiguities in the one-dimensional inverse scattering are also studied by Sabatier with the use of the Darboux-Bäcklund transformation. The nonuniqueness arises from the zero-energy poles, which are related to the value of the scattering matrix at zero energy. For the families of potentials considered here, we have \( T(k) = O(k^{N+M-1}), R(k) = \pm 1 + O(k) \), and \( L(k) = \pm 1 + O(k) \) as \( k \to 0 \). The nonuniqueness arises from the double or higher-order zeros of the transmission coefficient at \( k=0 \) or the unit value of the reflection coefficients at \( k=0 \) and specifying the ratio \( m_I(k, x)/m_r(k, x) \) at \( k=0, x=0 \) uniquely specifies the parameter and hence removes the nonuniqueness.

When the parameters \( a \) and \( \beta \) are nonnegative, the potentials considered here are positive everywhere and hence they do not support any bound states.

5. T. Aktosun, to be published.
10. T. Aktosun, to be published.
11. T. Aktosun, to be published.
15. A. Degasperis, to be published.