Chapter 2.2.4

INVERSE THEORY: PROBLEM ON THE LINE

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§1. Schrödinger Equation

Consider the one-dimensional Schrödinger equation

$$\frac{d^2\psi(k,x)}{dx^2} + k^2 \psi(k,x) = V(x) \psi(k,x), \quad x \in \mathbb{R},$$

with a real-valued potential $V$ belonging to $L^1_I(\mathbb{R})$, where $L^1_I(\mathbb{R})$ denotes the class of measurable potentials such that $\int_I dx (1 + |x|^p) |V(x)|$ is finite. The analysis of (1) is fundamental for the understanding of the direct and inverse scattering problems for many other related equations in one dimension. The direct scattering problem is the problem of finding appropriate solutions to (1) that can be used to describe the scattering process associated with the time-dependent Schrödinger equation. The solution of (1) allows one to identify a certain set of data, the scattering data, that describes characteristic features of the scattering process such as reflection and transmission. The inverse scattering problem, on the other hand, deals with the construction of $V$ using the scattering data. We refer to those solutions of (1) that behave like $e^{ikx}$ or $e^{-ikx}$ as $x \to \pm\infty$ for $k \in \mathbb{R} \setminus \{0\}$ as the scattering solutions. The bound states of (1) correspond to the square-integrable solutions, and they occur only at certain $k$ values on the positive imaginary axis in the complex plane $C$. The analysis of (1) at $k = 0$ requires a separate discussion.

There are several aspects of the inverse problem. The reconstruction problem is the problem of finding procedures to recover the potential from the scattering data. The characterisation problem is about the identification of necessary and sufficient conditions on the scattering data ensuring that the reconstructed potential belongs to a particular class. The stability problem consists of analysing how the constructed potential changes when the scattering data are perturbed. We will not discuss the stability problem here and refer the reader to Chadan and
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Sabatier (1989) and the references therein. Since inverse problems are in general ill posed, the aim is to find appropriate restrictions on the scattering data or on the potential so that the inverse scattering problem is well posed.

**Scattering Solutions and Scattering Coefficients**

Among the scattering solutions of (1) are the Jost solution from the left, $f_l$, and the Jost solution from the right, $f_r$, satisfying the boundary conditions

$$e^{-ikx}f_l(k,x) = 1 + o(1), \quad x \to +\infty, \quad (2)$$
$$e^{ikx}f_l(k,x) = ik + o(1), \quad x \to +\infty, \quad (2)$$
$$e^{-ikx}f_r(k,x) = 1 + o(1), \quad x \to -\infty, \quad (3)$$
$$e^{ikx}f_r(k,x) = -ik + o(1), \quad x \to -\infty, \quad (3)$$

where the prime denotes the derivative with respect to the spatial coordinate $x$. From the asymptotics

$$f_l(k,x) = \frac{e^{ikx}}{T(k)} + \frac{L(k)e^{-ikx}}{T(k)} + o(1), \quad x \to -\infty, \quad (4)$$
$$f_r(k,x) = \frac{e^{-ikx}}{T(k)} + \frac{R(k)e^{ikx}}{T(k)} + o(1), \quad x \to +\infty, \quad (5)$$

we obtain the scattering coefficients, namely, the transmission coefficient $T$, and the reflection coefficients $L$ and $R$ from the left and right, respectively. Since $V$ decays to 0 as $x \to \pm\infty$ in the sense of being in $L^1_0(\mathbb{R})$, the same transmission coefficient appears in (4) and (5).

We use $\mathbb{C}^*$ to denote the upper-half complex plane and $\mathbb{C}^* := \mathbb{C} \cup \mathbb{R}$; similarly, $\mathbb{C}^*$ denotes the lower-half complex plane and $\mathbb{C}^- := \mathbb{C} \cup \mathbb{R}$. Let $[f,g] := fg' - f'g$ denote the Wronskian of $f$ and $g$, and recall that the Wronskian of any two solutions of (1) is independent of $x$.

For each fixed $x \in \mathbb{R}$, $f_l(\cdot,x)$ and $f_r(\cdot,x)$ have extensions from $K \in \mathbb{R}$ to $K \in \mathbb{C}^*$ such that $f_l, f_r, f_l'$ and $f_r'$ are analytic in $K \in \mathbb{C}^*$ and continuous in $K \in \mathbb{C}^-$. The scattering coefficients can also be obtained from the Wronskian relations

$$2ik \frac{T(k)}{T(k)} = [f_l(k,x); f_l(k,x)], \quad k \in \mathbb{C}^-, \quad (6)$$
$$2ikL(k) \frac{T(k)}{T(k)} = [f_l(k,x); f_r(-k,x)], \quad k \in \mathbb{R}, \quad (7)$$
$$2ikR(k) \frac{T(k)}{T(k)} = [f_l(-k,x); f_r(k,x)], \quad k \in \mathbb{R}. \quad (8)$$

The left-hand side in (6) is analytic in $K \in \mathbb{C}^*$. In general, $f_l(\cdot,x)$ and $f_r(\cdot,x)$ do not have analytic extensions from $K$ to $\mathbb{C}^-$, and hence the reflection coefficients $L$ and $R$ cannot be extended as analytic functions from $K$ into $K$. However, if $V$ vanishes on the left half-line $\mathbb{R}^-$, then $f_l(\cdot,x)$ has an analytic extension to $\mathbb{C}^-$ and hence the left-hand side of (7) can be extended analytically to $\mathbb{C}^*$. Similarly, if $V$ vanishes on the right half-line $\mathbb{R}^+$, then $f_l(\cdot,x)$ has an analytic extension to $\mathbb{C}^-$ and hence the left-hand side of (8) can be extended analytically to $\mathbb{C}^*$.

In view of (2), (3), and the fact that $V$ is real valued, we have

$$f_l(-k^*,x) = f_l(k,x)^*, \quad f_r(-k^*,x) = f_r(k,x)^*, \quad k \in \mathbb{C}^*, \quad (9)$$

where the asterisk denotes complex conjugation. Consequently

$$T(-k^*) = T(k)^*, \quad R(-k^*) = R(k)^*, \quad (10)$$
$$L(-k) = L(k), \quad k \in \mathbb{R}. \quad (10)$$

We also have

$$R(k)T(k)^* = -L(k)^*T(k), \quad k \in \mathbb{R}, \quad (11)$$
$$|T(k)|^2 + |L(k)|^2 = 1 = |T(k)|^2 + |R(k)|^2, \quad k \in \mathbb{R}. \quad (12)$$

Thus, the scattering coefficients cannot exceed one in absolute value for real $K$. Furthermore, from (6) it follows that $T(k) \neq 0$ if $k \in \mathbb{R} \setminus \{0\}$, and hence the reflection coefficients are strictly less than 1 in absolute value when $k \in \mathbb{R} \setminus \{0\}$. For large $k$ we have

$$T(k) = 1 + O(1/k), \quad k \to \infty \text{ in } \mathbb{C}^-, \quad R(k) = O(1/k), \quad L(k) = O(1/k), \quad k \to \pm\infty. \quad (13)$$

**Bound States**

If $V \in L^1_0(\mathbb{R})$, then $T$ is meromorphic in $\mathbb{C}^*$ and the number of its poles is finite; each such pole is simple, occurs on the positive imaginary axis and corresponds to a bound state of $V$. Conversely, each bound state of $V$ corresponds to a pole of $T$ in $\mathbb{C}^*$. Let $N$ indicate the number of bound states, and use $k = i\kappa_j$ with $0 < \kappa_1 < \cdots < \kappa_N$ to denote the bound states. At the bound states the two Jost solutions are linearly dependent and

$$f_l(i\kappa_j,x) = f_r(i\kappa_j,x) = \gamma_j, \quad x \in \mathbb{R}, \quad (13)$$

for some real nonzero dependency constants $\gamma_j$. From (2) and (3) it is seen that

$$f_l(i\kappa_j,x) = e^{-i\kappa_j x}[1 + o(1)], \quad x \to +\infty, \quad (14)$$
$$f_r(i\kappa_j,x) = e^{i\kappa_j x}[1 + o(1)], \quad x \to -\infty, \quad (15)$$

and hence the bound-state solutions of (1) decay exponentially as $x \to \pm\infty$. The residue of $T$ at $k = i\kappa_j$ is found to be

$$\text{Res}(T,i\kappa_j) = i \left[ \int_{-\infty}^{\infty} dx f_l(i\kappa_j,x) f_r(i\kappa_j,x) \right]^{-1}. \quad (16)$$

The bound-state norming constants $c_d$ and $c_d'$ are defined as

$$c_d := \left[ \int_{-\infty}^{\infty} dx f_l(i\kappa_j,x)^2 \right]^{-1/2}, \quad (17)$$
$$c_d' := \left[ \int_{-\infty}^{\infty} dx f_r(i\kappa_j,x)^2 \right]^{-1/2}, \quad (17)$$

and they are related to the dependency constants via the residues of $T$ by
where \( \text{arg} \) denotes the continuous branch of the argument function normalised such that \( \text{arg} T(-\infty) = 0 \).

Here \( d = 1 \) if \( V \) is generic and \( d = 0 \) if \( V \) is exceptional. If \( V \in L^1(\mathbb{R}) \) but \( V \notin L^1_v(\mathbb{R}) \), then the poles of \( T \) may accumulate at \( k = 0 \) and hence the number of bound states may be infinite; this makes the recovery of \( V \) difficult, and in fact there are no general inversion methods for potentials in \( L^1(\mathbb{R}) \) using the scattering coefficients.

### Scattering Matrix

The scattering matrix is given by

\[
S(k) := \begin{bmatrix} T(k) & R(k) \\ L(k) & T(k) \end{bmatrix}, \quad k \in \mathbb{C},
\]

and it can be constructed in terms of the bound-state energies and either one of the reflection coefficients \( R \) and \( L \). Given \( R(k) \) for \( k \in \mathbb{R} \) and the bound-state poles \( k = i\kappa \), we can construct \( T \) and \( L \) as

\[
T(k) = \left( \prod_{j=1}^{N} \frac{k + i\kappa_j}{k - i\kappa_j} \right) \exp \left( \frac{1}{2\pi i} \int_{-\infty}^{\infty} dt \frac{\log(1 - |R(t)|^2)}{t - k - i0^+} \right),
\]

\[
k \in \mathbb{C}^*, \quad k \in \mathbb{C}^*,
\]

\[
L(k) = \frac{R(k)^* T(k)}{T(k)^*}, \quad k \in \mathbb{R}.
\]

Similarly, given \( L(k) \) for \( k \in \mathbb{R} \) and the bound-state poles \( k = i\kappa \), we can construct \( T \) and \( R \) as

\[
T(k) = \left( \prod_{j=1}^{N} \frac{k + i\kappa_j}{k - i\kappa_j} \right) \exp \left( \frac{1}{2\pi i} \int_{-\infty}^{\infty} dt \frac{\log(1 - |L(t)|^2)}{t - k - i0^+} \right),
\]

\[
k \in \mathbb{C}^*,
\]

\[
R(k) = \frac{L(k)^* T(k)}{T(k)^*}, \quad k \in \mathbb{R}.
\]

### Darboux Transformation

The Darboux transformation allows us to add to a given potential or to remove from it any number of bound states. It provides an alternative method to deal with bound states in the reconstruction of a potential.

Let us use a tilde to denote the quantities associated with the resulting Schrödinger equation when a bound state is added to \( (k) \) at \( k = i\kappa \) with \( \kappa > \kappa_N \) (with \( \kappa > 0 \) if 1 has no bound states). That is, \( \tilde{V} \) is the resulting potential, \( \tilde{T} \), \( \tilde{L} \) and \( \tilde{R} \) are the scattering coefficients, and \( \tilde{f}_1 \) and \( \tilde{f}_2 \) are the Jost solutions from the left and from the right, respectively. We have

\[
\tilde{V}(x; \kappa, \alpha) = V(x) - 2\xi'(x; \kappa, \alpha),
\]

\[
\tilde{f}_1(k, x; \kappa, \alpha) = \frac{1}{i(k + i\kappa)} \left[ f_1^*(k, x) - \chi(x; \kappa, \alpha) f_1(k, x) \right],
\]

where \( \tilde{V} \) is the Schrödinger operator with the new potential \( \tilde{V} \). The Jost solutions \( f_1 \) and \( f_2 \) are given by

\[
\begin{align*}
\tilde{f}_1 &= \frac{1}{i(k + i\kappa)} \left[ f_1^*(k, x) - \chi(x; \kappa, \alpha) f_1(k, x) \right], \\
\tilde{f}_2 &= f_2.
\end{align*}
\]
\[ f_i(k;x;k,\alpha) = \frac{i}{k+i\kappa} \left[ f_i'(k,x) - \chi(x;k,\alpha) f_i(k,x) \right], \quad (33) \]
\[ T_i(k;x;k,\alpha) = \frac{k+i\kappa}{k-i\kappa} T_i(k), \quad (34) \]
\[ \tilde{T}_i(k;x;k,\alpha) = -\frac{k+i\kappa}{k-i\kappa} L_i(k), \quad \tilde{R}_i(k;x;k,\alpha) = -\frac{k+i\kappa}{k-i\kappa} R_i(k), \quad (35) \]
where
\[ \chi(x;k,\alpha) = \frac{f_i'(ik\kappa;x;k,\alpha)}{f_i'(ik\kappa;x;k,\alpha)} + \alpha f_i'(ik\kappa;x;k,\alpha), \quad x \in \mathbb{R}, \quad (36) \]
and \( \alpha \) corresponds to the dependency constant at the bound state \( k = i\kappa \), i.e.,
\[ \alpha = \frac{f_i'(ik\kappa;x;k,\alpha)}{f_i'(ik\kappa;x;k,\alpha)}. \quad (37) \]
The function \( \chi(x;k,\alpha) \) belongs to \( L^1_1(\mathbb{R}) \), and hence \( \tilde{V} \in L^1_1(\mathbb{R}) \) whenever \( V \in L^1_1(\mathbb{R}) \).

Conversely, assume that \( \tilde{V} \) is real valued and belongs to \( L^1_1(\mathbb{R}) \) and that its lowest bound-state energy corresponds to \( k = i\kappa \) for some \( \kappa > 0 \). Let \( f_i \) and \( f_i' \) denote the Jost solutions for \( \tilde{V} \), from the left and from the right, respectively. After the removal of the bound state at \( k = i\kappa \), let us denote the resulting potential by \( V \) with the corresponding Jost solutions \( f_i \) and \( f_i' \). Then we get
\[ V(x) = \tilde{V}(x) - 2\eta(x), \]
\[ f_i(k,x) = \frac{1}{ik(k-i\kappa)} \left[ f_i'(k,x) - \eta(x) f_i(k,x) \right], \]
\[ f_i'(k,x) = \frac{i}{k-i\kappa} \left[ f_i'(k,x) - \eta(x) f_i'(k,x) \right], \]
where \( \eta(x) : = f_i'(ik\kappa;x;k,\alpha) f_i'(ik\kappa;x;k,\alpha) \).

In recovering \( V \) from the scattering data, it is possible to remove all the bound states from the data first and construct the resulting potential \( V^{(0)} \) corresponding to the scattering coefficients \( T^{(0)}, R^{(0)}, L^{(0)} \). Then the bound-state information can be used as in (31)–(37) to construct \( V \). We have
\[ T(k) = T^{(0)}(k) \prod_{i=1}^{N} \frac{k+i\kappa_i}{k-i\kappa_i}, \quad (38) \]
\[ L(k) = (-1)^N L^{(0)}(k) \prod_{i=1}^{N} \frac{k+i\kappa_i}{k-i\kappa_i}, \quad (39) \]
\[ R(k) = (-1)^N R^{(0)}(k) \prod_{i=1}^{N} \frac{k+i\kappa_i}{k-i\kappa_i}. \]
The potential \( V^{(0)} \) belongs to \( L^1_1(\mathbb{R}) \) whenever \( V \in L^1_1(\mathbb{R}) \).

### Fragmentation of the Potential

By partitioning the real axis as \( \mathbb{R} = \cup_{j=1}^{p} (x_{j-1},x_j) \) with \( x_0 := -\infty, x_p := +\infty, \) and \( x_{j-1} < x_j \) for \( j = 1, \ldots, p \), we obtain a fragmentation of the potential as \( V(x) = \sum_{j=1}^{p} V_j(x) \), where
\[ V_j(x) := \begin{cases} V(x), & x \in (x_{j-1},x), \\ 0, & \text{elsewhere.} \end{cases} \]

Let \( N_j, T_j, R_j \) and \( L_j \) denote the number of bound states, the transmission coefficient and the reflection coefficients from the right and left, respectively, for the potential \( V_j \). The number of bound states for \( V \) satisfies, see, e.g., (Aktosun et al. 1998b),
\[ 1 - p + \sum_{j=1}^{p} N_j \leq N \leq \sum_{j=1}^{p} N_j, \]
and the scattering coefficients for \( V \) can be expressed in terms of those for \( V_j \) as
\[ \begin{bmatrix} 1 \\ T(k) \\ R(k) \\ L(k) \\ T^{-1}(k) \end{bmatrix} = \begin{bmatrix} 1 \\ T_1(k) \\ R_1(k) \\ L_1(k) \\ T_1^{-1}(k) \end{bmatrix} \cdots \begin{bmatrix} 1 \\ T_p(k) \\ R_p(k) \\ L_p(k) \\ T_p^{-1}(k) \end{bmatrix}, \quad k \in \mathbb{R}. \quad (40) \]

Factorisations of the form (40) are helpful in the recovery of potentials that are partially known.

### §2. Methods to Solve the Inverse Scattering Problem

When there are no bound states, either one of the reflection coefficients \( R \) and \( L \) uniquely determines a real-valued potential \( V \) in \( L^1_1(\mathbb{R}) \). However, when there are bound states, for the unique determination of \( V \), in addition to one reflection coefficient and the bound-state energies, we must also specify the bound-state norming constant or, equivalently, the dependency constant for each bound state. As our scattering data we can choose either the left scattering data \( R_s(k), \{\kappa_i\} \) or the right scattering data \( R_r(k), \{\kappa_i\} \). These two are equivalent to each other, and from (18) and (26)–(30) it is seen that each is equivalent to \( \{S, \{\eta_j\}\} \).

### Riemann–Hilbert Problems

Since \( k \) appears as \( k^2 \) in (1), the functions \( f_i(-k,x) \) and \( f_i(-k,x) \) are also solutions of (1) and they can be expressed as linear combinations of the Jost solutions \( f_i(k,x) \) and \( f_i(k,x) \) for real \( k \) as
\[ f_i(-k,x) = T(k)f_i(k,x) - R(k)f_i(k,x), \quad k \in \mathbb{R}, \]
\[ f_i(-k,x) = T(k)f_i(k,x) - L(k)f_i(k,x), \quad k \in \mathbb{R}, \]
or equivalently as
\[ m_i(-k,x) = T(k)m_i(k,x) - R(k)e^{2ikx} m_i(k,x), \quad k \in \mathbb{R}, \]
\[ m_i(-k,x) = T(k)m_i(k,x) - L(k)e^{-2ikx} m_i(k,x), \quad k \in \mathbb{R}, \]
where \( m_i \) and \( m_r \) are the Faddeev functions defined as
\[ m_i(k,x) := e^{-ikx} f_i(k,x), \quad m_r(k,x) := e^{ikx} f_i(k,x). \quad (43) \]
Each of (41) and (42) can be viewed as a Riemann–Hilbert problem (Newton, 1983; Ablowitz and Clarkson, 1991) where, given the scattering coefficients for $k \in \mathbb{R}$, the aim is to construct $m_1$ and $m_2$ such that, for each $x \in \mathbb{R}$, $m_1(\cdot, x)$ and $m_2(\cdot, x)$ are analytic in $\mathbb{C}^+$, are continuous in $\mathbb{C}$ and behave like $1 + O(1/k)$ as $k \to \infty$ in $\mathbb{C}^+$.

**Faddeev–Marchenko Method**

In this method (Faddeev, 1967; Deift and Trubowitz, 1979; Marchenko, 1982; Chadan and Sabatier, 1989) the potential is constructed from the left scattering data $\{R_1(\cdot, \kappa), \{c_{ij}\}\}$ by solving the left Marchenko integral equation or from the right scattering data $\{L_1(\cdot, \kappa), \{c_{ij}\}\}$ by solving the right Marchenko integral equation.

The left Marchenko equation using the left scattering data as the input is given by

$$B_l(x, \alpha) = g_l(2x + \alpha) + \int_{0}^{\infty} dp \; g_l(2x + \alpha + \beta) B_l(x, \beta), \quad \alpha > 0,$$

(44)

where

$$g_l(\alpha) := \tilde{R}(\alpha) - \sum_{i=1}^{N} c_{ij}^l e^{-\kappa_{ij} \alpha},$$

(45)

with

$$\tilde{R}(\alpha) := \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \; R(k) e^{ika}.$$  

(46)

We can obtain (44) from (41) by using a Fourier transformation. The potential is obtained from the solution $B_l(x, \alpha)$ as

$$V(x) = -2 \frac{dB_l(x, 0^+)}{dx},$$

(47)

and the Jost solution from the left is constructed as

$$f_l(k, x) = e^{ikx} \left[ 1 + \int_{0}^{\infty} d\alpha \; B_l(x, \alpha) e^{i\kappa\alpha} \right].$$

(48)

Similarly, via a Fourier transformation of (42), using the right scattering data as the input we obtain the right Marchenko equation

$$B_r(x, \alpha) = g_r(-2x + \alpha) + \int_{0}^{\infty} d\beta \; g_r(-2x + \alpha + \beta) B_r(x, \beta), \quad \alpha > 0,$$

(49)

where

$$g_r(\alpha) := \tilde{L}(\alpha) - \sum_{i=1}^{N} c_{ji}^r e^{-\kappa_{ij} \alpha},$$

(50)

with

$$\tilde{L}(\alpha) := \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \; L(k) e^{ika}.$$  

(51)

The potential is recovered by using

$$V(x) = \frac{2}{dx} \frac{dB_r(x, 0^+)}{dx},$$

(52)

and the Jost solution from the right is constructed as

$$f_r(k, x) = e^{ikx} \left[ 1 + \int_{0}^{\infty} d\alpha \; B_r(x, \alpha) e^{i\kappa\alpha} \right].$$

(53)

The Marchenko equations are often written in a different form. Letting

$$K_+(x, y) := B_l(x, y - x), \quad c_{ij}^+: = c_{ij}^l,$$

$$M_+(y) := -g_l(y) = \tilde{R}(y) + \sum_{i=1}^{N} c_{ij}^l e^{-\kappa_{ij} y},$$

the left Marchenko equation (44) takes the form

$$K_+(x, y) + M_+(y + x) + \int_{x}^{\infty} dz M_+(y + z) K_+(x, z) = 0, \quad y > x.$$  

Once this Marchenko equation is solved for $K_+(x, y)$, the potential is obtained as

$$V(x) = -2 \frac{dK_+(x, x^+)}{dx},$$

and the Jost solution from the left is constructed as

$$f_l(k, x) = e^{ikx} \left[ 1 + \int_{0}^{\infty} d\alpha \; B_l(x, \alpha) e^{i\kappa\alpha} \right].$$

Similarly, by letting

$$K_-(x, y) := B_r(x, y - x), \quad c_{ij}^-: = c_{ij}^r,$$

$$M_-(y) := -g_r(y) = \tilde{L}(y) + \sum_{i=1}^{N} c_{ji}^r e^{-\kappa_{ij} y},$$

the right Marchenko equation (49) takes the form

$$K_-(x, y) + M_-(y + x) + \int_{x}^{\infty} dz M_-(y + z) K_-(x, z) = 0, \quad y < x.$$  

Once this Marchenko equation is solved for $K_-(x, y)$, the potential is obtained as

$$V(x) = -2 \frac{dK_-(x, x^+)}{dx},$$

and the Jost solution from the right is constructed as

$$f_r(k, x) = e^{-ikx} \left[ 1 + \int_{0}^{\infty} d\alpha \; B_r(x, \alpha) e^{-i\kappa\alpha} \right].$$

**Gel'fand–Levitan Method**

In this method (Newton, 1983; Levitan, 1987) the potential is constructed from the corresponding spectral function, which is a $2 \times 2$ real-valued matrix function of energy $E = k^2$.

The Gel'fand–Levitan equation is given by

$$b(x, \alpha) = \nu(\alpha, x) - \int_{-\infty}^{\infty} db \; \nu(\alpha, b) b(x, \beta), \quad -|x| < \alpha < |x|,$$

where

$$\nu(\alpha, \beta) := \int_{-\infty}^{\infty} dE \; \left[ e^{-ik\alpha} \left( \frac{e^{ik\beta}}{E - \beta} \right) \right] dE,$$

(54)
with the (modified) spectral function given as

\[
\frac{d\rho}{dE} = \begin{cases} \frac{1}{4\pi k} \left[(J(k)j(k))^{-1} - I_2\right], & E > 0, \\ \sum_{j=1}^N M_j \delta(E - E_j), & E < 0. \end{cases}
\]

Here \(I_2\) is the \(2 \times 2\) identity matrix, the dagger denotes the matrix adjoint, \(\delta\) indicates the Dirac delta distribution, \(M_j\) are certain \(2 \times 2\) constant matrices related to the bound-state data and \(J(k)\) is the Jost matrix, which is a \(2 \times 2\) matrix-valued function of \(k\) relating the Jost solutions \(f_i\) and \(f_i^{\prime}\) and the regular solutions \(\phi_i\) and \(\phi_i^{\prime}\) as

\[
T(k)J(k)\begin{bmatrix} f_i(k,x) \\ f_i^{\prime}(k,x) \end{bmatrix} = \begin{bmatrix} \phi_i(k,x) \\ \phi_i^{\prime}(k,x) \end{bmatrix},
\]

where \(\phi_i\) and \(\phi_i^{\prime}\) are the solutions of (1) satisfying

\[
\phi_i(k,0) = 1, \quad \phi_i^{\prime}(k,0) = ik,
\]

\[
\phi_i(k,0) = 1, \quad \phi_i^{\prime}(k,0) = -ik.
\]

In the Gel'fand–Levitan method, if we want to use as an input either the left or right scattering data instead of the spectral function, then we first need to construct \(J(k)\) and \(M_j\) from such data; the reader is referred to Newton (1983) for such a construction.

The potential is obtained from the solution \(b(x,\alpha)\) of the Gel'fand–Levitan equation as

\[
V(x) = -2 \frac{db(x,x;\alpha)}{dx}, \quad \pm x \geq 0,
\]

and the regular solutions are constructed as

\[
\begin{bmatrix} \phi_i(k,x) \\ \phi_i^{\prime}(k,x) \end{bmatrix} = \begin{bmatrix} e^{ikx} \\ e^{-ikx} \end{bmatrix} - \int_x^\infty da \cdot b(x,\alpha) \begin{bmatrix} e^{ik\alpha} \\ e^{-ik\alpha} \end{bmatrix}.
\]

**Singular Integral Equations**

From (13), (18), and (41), with the help of a contour integration, we find that the Faddeev function \(m_l\) satisfies the singular integral equation

\[
m_l(k,x) = 1 - \sum_{j=1}^N \frac{e^{2ikx}}{k + ik_j} m_l(k,x),
\]

where the left scattering data \(\{R_j, \{k_j\}, \{c_{ij}\}\}\) are used as the input. Solving this singular integral equation for \(m_l(k,x)\), the potential can be obtained with the help of the Schrödinger equation by using

\[
V(x) = \frac{1}{m_l(k,x)} [m_l^2(k,x) + 2ikm_l(k,x)].
\]
Newton–Marchenko Method

As mentioned earlier, knowledge of \( \{S, \{\gamma_i\}\} \) is equivalent to that of either the left or the right scattering data. In the Newton–Marchenko method the potential is constructed from the scattering data consisting of the whole scattering matrix and the bound-state dependency constants. The dependency constants are used to fix the norms of the eigenvectors of the Jost matrix at the bound-state energies. The reader is referred to Newton (1983) and Chadan and Sabatier (1989) for details. This method has a generalisation to higher dimensions because it uses the scattering matrix itself as an input to the Newton–Marchenko integral equation.

When there are no bound states, the Newton–Marchenko equation is the coupled system given by

\[
\eta(x, \alpha) = H(x, \alpha) \left[ \begin{array}{c} 1 \\ 1 \end{array} \right] + \int_0^\infty d\beta H(x, \alpha + \beta) \eta(x, \beta), \quad \alpha > 0,
\]

where

\[
H(x, \alpha) := \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \begin{bmatrix} T(k) - 1 & L(k) e^{2i\alpha k} \\ R(k) e^{2i\alpha k} & T(k) - 1 \end{bmatrix} e^{-ikx}.
\]

We can obtain (60) from (41) and (42) by using \( k \mapsto -k \), a multiplication by \( T(k) \) and a Fourier transformation in succession. Once the coupled system (60) is solved, we have the Faddeev functions by using

\[
\begin{bmatrix} T(k) m_l(k, x) \\ T(k) m_r(k, x) \end{bmatrix} = \left[ \begin{array}{c} 1 \\ 1 \end{array} \right] + \int_0^\infty d\alpha \eta(x, \alpha) e^{ikx},
\]

and the potential is obtained as

\[
V(x) = -2 \frac{\partial \eta(x, 0^+)}{\partial x} \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] = 2 \frac{\partial \eta(x, 0^+)}{\partial x} \left[ \begin{array}{c} 1 \\ 0 \end{array} \right].
\]

When there are bound states, they can first be removed by using Darboux transformations; after the inverse scattering problem is solved by the Newton–Marchenko method, the modifications due to the bound states can be implemented (Newton, 1983).

Wiener–Hopf Factorisation

In this method we obtain the Wiener–Hopf factorisation of the unitarily dilated scattering matrix given by

\[
G(k, x) := \begin{bmatrix} T(k) & -R(k) e^{2i\alpha k} \\ -L(k) e^{-2i\alpha k} & T(k) \end{bmatrix}, \quad k \in \mathbb{R}.
\]

Such a factorisation has the form

\[ G(k, x) = G_-(k, x) D(k) G_+(k, x), \quad k \in \mathbb{R}, \]

where

\[ D(k) = \begin{bmatrix} k - i & \rho_1 \\ k + i \end{bmatrix} Q_+ + \begin{bmatrix} k - i \\ k + i \end{bmatrix} \rho_2 Q_- \]

For each \( x \in \mathbb{R} \), the matrix function \( G_\pm(\cdot, x) \) and its inverse \( G_\pm(\cdot, x)^{-1} \) are continuous in \( \mathbb{C}^2 \) and analytic in \( \mathbb{C}^2 \). Moreover, \( G_\pm(k, x) \rightarrow I_2 \) as \( k \rightarrow \pm \infty \) in \( \mathbb{C}^2 \). The numbers \( \rho_1 \) and \( \rho_2 \) are integers called partial indices and are uniquely determined by \( G(k, x) \); if both \( \rho_1 \) and \( \rho_2 \) are zero, the factorisation is canonical, otherwise noncanonical. The matrices \( Q_\pm \) are complementary rank-one projections and can be chosen as

\[
Q_\pm := \frac{1}{2} \begin{bmatrix} 1 & \pm i \\ \pm i & 1 \end{bmatrix}.
\]

Note that the system (41) and (42) can be written as

\[
\begin{bmatrix} m_l(-k, x) \\ m_r(-k, x) \end{bmatrix} = G(k, x) \begin{bmatrix} m_l(k, x) \\ m_r(k, x) \end{bmatrix}, \quad k \in \mathbb{R}.
\]

Hence, once the Wiener–Hopf factorisation of \( G(k, x) \) is obtained, we have essentially solved the inverse scattering problem. In the exceptional case without bound states we have \( D(k) = I_2 \), and the Faddeev functions \( m_l \) and \( m_r \) are obtained from \( G_-(k, x) \) as

\[
\begin{bmatrix} m_l(k, x) \\ m_r(k, x) \end{bmatrix} = G_-(k, x) \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad k \in \mathbb{C}^2.
\]

In the generic case without bound states we have \( \rho_1 = 0 \) and \( \rho_2 = 1 \). The Wiener–Hopf factors \( G_\pm(k, x) \) depend on a free parameter \( a \in [0, +\infty) \), but that free parameter does not appear in the Faddeev functions \( m_l \) and \( m_r \), which are recovered as

\[
\begin{bmatrix} m_l(k, x) \\ m_r(k, x) \end{bmatrix} = G_-(k, x) \begin{bmatrix} Q_+ \left( \frac{k + i}{k} \right) Q_- \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad k \in \mathbb{C}^2.
\]

Having found the Faddeev functions, the potential is obtained as in (57) or (59).

In the exceptional case, when there are \( N \) bound states, the partial indices are given by \( \rho_1 = \rho_2 = -N \). The Faddeev functions corresponding to the potential \( V^{[0]} \) with the scattering coefficients \( T^{[0]} \) and \( R^{[0]} \) and \( L^{[0]} \) as in (38) and (39) are obtained as

\[
\begin{bmatrix} m_l^{[0]}(k, x) \\ m_r^{[0]}(k, x) \end{bmatrix} = (\prod_{\nu=1}^{N} k + i \gamma_{\nu}) G_-(k, x) \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad k \in \mathbb{C}^2.
\]

Even though \( G_-(k, x) \) depends on \( 2N \) parameters, the Faddeev functions in (62) depend on \( N \) parameters that are uniquely determined by the \( N \) normalisation constants. The potential \( V \) and the Jost solutions can be constructed as in (31)–(33) to complete the solution of the inverse scattering problem.

In the generic case, when there are \( N \) bound states, the partial indices are given by

\[
\rho_1 = \begin{cases} -N, & \text{even}, \\ -N+1, & \text{odd} \end{cases}, \quad \rho_2 = \begin{cases} -N+1, & \text{even}, \\ -N, & \text{odd} \end{cases}.
\]
The Faddeev functions corresponding to the potential \( V^{(0)} \) are, for \( k \in \mathbb{C} \), obtained as
\[
\begin{bmatrix}
m_1^{(0)}(k,x) \\
m_2^{(0)}(k,x)
\end{bmatrix}
= \begin{cases}
\left( \prod_{r=1}^{N} \frac{k + \text{ik}_r}{k - \text{ik}_r} \right) G_r(-k,x) \left[ Q_r + \left( \frac{\text{ik}_r}{k} \right) Q_r \right] \left[ 1 \right], & \text{N even}, \\
\left( \prod_{r=1}^{N} \frac{k + \text{ik}_r}{k - \text{ik}_r} \right) G_r(-k,x) \left[ \left( \frac{\text{ik}_r}{k} \right) Q_r + Q_r \right] \left[ 1 \right], & \text{N odd}.
\end{cases}
\]
(62)

Even though \( G_r(-k,x) \) depends on \( 2N + 1 \) parameters, the Faddeev functions in (62) depend on \( N \) parameters that are uniquely determined by the \( N \) normalisation constants. Again, the potential \( V \) and the Jost solutions can be constructed as in (31)–(33) to complete the solution of the inverse scattering problem.

In fact, the Wiener–Hopf factorisation of \( G(k,x) \) yields not only the potential \( V \) corresponding to the scattering matrix \( S \), but also the potential(s) corresponding to the scattering matrix obtained by reversing the signs of the reflection coefficients. In the generic case such potentials do not belong to \( L^1(\mathbb{R}) \), but they are closely related to that class, as indicated in §4. The additional constants appearing in the Wiener–Hopf factors \( G_\pm(k,x) \), in which there are \( N \) such factors in the exceptional case and \( N + 1 \) of them in the generic case, can be fixed by using the additional normalisation constants related to the potential(s) corresponding to the scattering matrix obtained by changing the signs of the reflection coefficients. A detailed investigation of these potentials can be found in Chadan and Sabatier (1989) and Akhtosum et al. (1993).

### D-Bar Method

An elementary treatment of the D-bar method in inverse scattering can be found in Chadan and Sabatier (1989) and Ablowitz and Clarkson (1991). Let us use \( k_R \) and \( k_1 \) to denote the real and imaginary parts, respectively, of the complex variable \( k \), and use \( \bar{k} \) for the complex conjugate of \( k \). The D-bar derivative of a function \( g \) is defined as
\[
\frac{\partial g}{\partial \bar{k}} := \frac{1}{2} \left( \frac{\partial g}{\partial k_R} + i \frac{\partial g}{\partial k_1} \right),
\]
and it measures the “departure from analyticity” because \( \partial g / \partial \bar{k} \equiv 0 \) in domains where \( g \) is analytic in \( k \). For example, we have
\[
\frac{\partial}{\partial \bar{k}} \left( \frac{1}{k-k_0} \right) = \pi \delta(k-k_0),
\]
(63)
where \( \delta \) denotes the Dirac delta function on the complex plane. The Gauss–Green formula generalises the Cauchy integral formula, and for \( g(k) \) vanishing at \( \infty \) it provides the integral representation
\[
g(k) = \frac{1}{2 \pi i} \oint_D \frac{dz}{z-k} g(z) + \frac{1}{2 \pi i} \oint_D \frac{dz \wedge \bar{z} \wedge \partial g(z)}{dz \wedge \bar{z} \wedge \partial \bar{z}},
\]
(64)
where the second integral is over a domain \( D \), the first integral is along the positively oriented boundary \( \partial D \) of \( D \), and \( dz \wedge \bar{z} \wedge \partial g(z) = -2i \partial z \wedge dz \wedge d\bar{z} \). The idea behind the D-bar method is to relate \( g \) and \( \partial g / \partial k \) to the scattering data so that (64) becomes an integral equation that can be solved for \( g \). The variable \( x \) appears in \( g \) as a parameter, and the potential \( V \) can be recovered from the solution \( g \).

For example, let
\[
g(k) = \begin{cases}
T(k)m_r(k,x) - 1, & k_1 > 1, \\
m_r(-k,x) - 1, & k_1 \leq 0,
\end{cases}
\]
(65)
where we have suppressed the dependence of \( g \) on \( x \). With the help of (38) and (63), we can evaluate the D-bar derivative of \( g \) as
\[
\frac{\partial g(k)}{\partial k} = \sum_{i=1}^{N} \text{Res}(T, ik_i) m_i(ik_i, x) \delta(k - ik_i) + [T(k) m_r(k,x) - m_r(-k,x)] \delta(k_1).
\]
(66)
Using (13), (18), (41) and (43), we can write (66) as
\[
\frac{\partial g(k)}{\partial k} = \sum_{i=1}^{N} c_i^2 e^{-2ikx} m_i(ik_i, x) \delta(k - ik_i)
\]
(67)
Thus, from (65) and (67) we get the D-bar equation
\[
\frac{\partial g(k)}{\partial k} = \begin{cases}
\frac{i}{2} R(k) e^{2ikx} \delta(k_1) - [\sum_{i=1}^{N} c_i^2 e^{-2ikx} g(ik_i)] \\
x[g(k) + 1], & k \in \mathbb{C},
\end{cases}
\]
(68)
with the boundary condition that \( g(k) \to 0 \) as \( k \to \infty \) in \( \mathbb{C} \). Note that by choosing \( D \) as the entire complex plane (and hence the boundary \( \partial D \) as the infinite circle centred at the origin) in (64) and using (67) in (64), we obtain the singular integral equation given in (56). Once \( g \), and hence also the Faddeev function \( m_r \), is obtained, \( V \) can be recovered as in (57).

In a similar manner, letting
\[
g(k) = \begin{cases}
T(k)m_r(k,x) - 1, & k_1 > 1, \\
m_r(-k,x) - 1, & k_1 \leq 0,
\end{cases}
\]
we obtain the D-bar equation
\[
\frac{\partial g(k)}{\partial k} = \begin{cases}
\frac{i}{2} L(k) e^{-2ikx} - [\sum_{i=1}^{N} c_i^2 e^{2ikx} g(ik_i)] \\
x[g(-k) + 1], & k \in \mathbb{C},
\end{cases}
\]
(68)
with the boundary condition that \( g(k) \to 0 \) as \( k \to \infty \) in \( \mathbb{C} \). Note that using (68) in (64) we obtain (58). Once (68) is solved for \( g \), we can recover \( V \) as in (59).
Other Methods

For various other methods, such as time-domain methods, various iterative methods like layer stripping and various approximation methods like Born approximation, the reader is referred to Chadan and Sabatier (1989) and the references therein.

§3. Characterisation

A characterisation for a specific class of potentials is to present some necessary and sufficient conditions on the scattering data that guarantee that there exists a corresponding unique potential in that class. Such conditions are usually obtained by using the Faddeev–Marchenko method. The characterisation conditions can be stated for the left scattering data, for the right scattering data or for the combination of both. For a characterisation in the class of real-valued potentials belonging to \( L^1_1(\mathbb{R}) \), the reader is referred to Deift and Trubowitz (1979). Some characterisations in the class of real-valued potentials belonging to \( L^1_1(\mathbb{R}) \) were given by Melin (1985) and Marchenko (1986).

Given the scattering data with the reflection coefficient \( R(k) \) for \( k \in \mathbb{R} \), the bound-state energies \(-\kappa_i^2\) with \( 0 < \kappa_1 < \ldots < \kappa_N \) and the bound-state norming constants \( q_i \), the following conditions (Aktosun and Klaus, 2000) form a characterisation in the class of real-valued potentials in \( L^1_1(\mathbb{R}) \):

(i) \( R \) is continuous on \( \mathbb{R} \), and \( R(-k) = R(k)^* \) for \( k \in \mathbb{R} \).
(ii) \( |R(k)| \leq 1 - Ck^2 / (1 + k^2) \) on \( \mathbb{R} \) for some constant \( C > 0 \).
(iii) \( R(0) \in [-1, 1] \).
(iv) \( R(k) = O(1/k) \) as \( k \to \pm \infty \).
(v) The function \( k / T(k) \), where \( T(k) \) is given by (27), is continuous in \( \mathbb{C}^+ \).
(vi) The functions \( \tilde{R} \) and \( \tilde{L} \) defined in (46) and (51), respectively, where \( L(k) \) is obtained from (28), are absolutely continuous. Moreover, \( \tilde{R} \in L^1_1(\mathbb{R}, +\infty) \) and \( \tilde{L} \in L^1_1(-\infty, a) \) for any \( a \in \mathbb{R} \).

When these conditions are satisfied, both the left and right Marchenko equations are uniquely solvable and the right-hand sides of (47) and (52) are equal. Thus, \( V \) can be obtained from either (47) or (52); moreover, \( V \) belongs to \( L^1_1(\mathbb{R}) \).

We can state the characterisation conditions differently, especially without any reference to the continuity at \( k = 0 \). For example, in (iv) we can replace \( O(1/k) \) by \( O(1/k) \) because of (vi). Also, as in Marchenko (1986), we can demand (i) and (ii) only for \( k \in \mathbb{R} \setminus \{0\} \) and simultaneously replace (iii) with the condition that \( k[1 + R(k)] / T(k) \to 0 \) as \( k \to 0 \) on \( \mathbb{R} \).

§4. Special Cases

When the potential is partially known, it may be possible to recover the unknown part of the potential without knowledge of the full scattering data. For example, we can use (40) to obtain the scattering data associated with the unknown part of the potential. There are also other approaches to such an inverse problem; see, for example, Novikova and Markushevich (1987), Aktosun et al. (1993), Aktosun (1994, 1996), Rundell and Sacks (1994), Grébert and Weder (1995), Gesztesy and Simon (1997) and the references therein.

Potentials Vanishing on a Half-Line

Assume the potential \( V \) of (1) is real valued, \( V(x) = 0 \) for \( x > 0 \) and \( V \in L^1_1(\mathbb{R}^-) \). Then the corresponding reflection coefficient \( R \) has a meromorphic extension to \( \mathbb{C}^+ \) with simple poles coinciding exactly with the poles of \( T \). Thus, the bound-state energies are uniquely determined once \( R(k) \) is known on any nontrivial interval on the real axis. Moreover, the bound-state norming constants are determined by the residues of \( R \) at such poles from \( c_i^2 = -i \text{Res}(R, ik) \). Hence, \( V \) is uniquely determined by \( R \) alone without specifying any bound-state information. However, \( L \) alone cannot uniquely determine \( V \), but \( \{L, \{\kappa_i\}\} \) does.

A similar result holds if \( V \) is real valued, \( V(x) = 0 \) for \( x < 0 \) and \( V \in L^1_1(\mathbb{R}^+) \). Then \( V \) is uniquely determined by \( L \) alone without specifying any bound-state information. However, \( R \) alone cannot uniquely determine \( V \), but \( \{R, \{\kappa_i\}\} \) does.

Rational Scattering Coefficients

When the scattering coefficients are rational functions of \( k \) in \( \mathbb{C} \), the corresponding potentials are known as Bargmann potentials. Such potentials decay exponentially as \( x \to \pm \infty \), or they may decay exponentially at one end of the real axis and vanish on the opposite half-line. These potentials can be obtained algebraically with the help of (36), (58) and a contour integration. We refer the reader to Chadan and Sabatier (1989) for the details of this method. Some methods from the theory of realisations of rational matrix functions (Alpay and Gohberg, 1998; van der Mee, 2000) can also be used to obtain closed-form expressions for the potential.

Potentials with Dirac Delta Distributions

Assume that to the real potential \( V \) belonging to \( L^1_1(\mathbb{R}) \), the Dirac delta distributions are added at some points so that the resulting potential \( W \) is given by

\[
W(x) = V(x) + \sum_{j=1}^{n} b_j \delta(x - x_j),
\]

We can require the continuity in (v) only in \( C^{-1} \).
where \( b_j \) are some real nonzero constants. Instead of vanishing as \( o(1/k) \) as \( k \to \pm \infty \), the reflection coefficients gain some \( O(1/k) \) terms such that

\[
2ik R(k) = \sum_{j=1}^{n} b_j e^{-2ikx_j} + o(1), \quad k \to \pm \infty.
\]

\[
2ik L(k) = \sum_{j=1}^{n} b_j e^{2ikx_j} + o(1), \quad k \to \pm \infty.
\]

Apart from some minor modifications, the inversion methods indicated in §2 remain applicable to recover \( W \).

**Potentials Obtained When the Signs of Reflection Coefficients Are Changed**

If we reverse the signs of the reflection coefficients in (26) corresponding to \( V \in L_1^2(R) \) without changing the sign of the transmission coefficient, it is possible to characterise (Aktosun et al., 1993) the corresponding potentials. Such potentials have the form

\[
U(x) = \frac{2e_+}{x^2 + 1} \theta(x) + \frac{2e_-}{x^2 + 1} \theta(-x) + Q(x),
\]

where \( \theta(x) \) is the Heaviside function, \( Q \) is some potential belonging to \( L_1^2(R) \) and \( e_\pm \in (0, 1) \). The exceptional case occurs if and only if \( e_+ = e_- = 0 \); generically \( U \notin L_1^2(R) \). In the generic case, one or both of the corresponding Jost solutions blow up like \( O(1/k) \) as \( k \to 0 \), and one additional normalisation constant at zero energy, in addition to the \( N \) constants for the \( N \) bound states, must be specified to determine such potentials uniquely.

**§5. Steplike Potentials**

Consider (1) when the potential has different constant asymptotics as \( x \to \pm \infty \). Without any loss of generality, it is enough to consider the case \( V(x) \to 0 \) as \( x \to -\infty \) and \( V(x) \to c^2 \) as \( x \to +\infty \) for some \( c > 0 \). To be precise, assume that \( V \) is real valued and satisfies \( V \in L_1^2(R^-) \) and \( V - c^2 \in L_1^2(R^+) \). The reader is referred to Buslaev and Fomin (1962) and Cohen and Kappeler (1985) for details. Since \( V \) has different asymptotics as \( x \to \pm \infty \), the transmission coefficients from the left and from the right, \( T_l \) and \( T_r \), are unequal but related by

\[
T_r(k) = \frac{\xi}{k} T_l(k), \quad (69)
\]

where \( \xi := \sqrt{k^2 - c^2} \), using the branch of the (complex) square-root function with \( \text{Im} \xi \geq 0 \). The mapping \( k \mapsto \xi \) is analytic from \( C^* \) to itself and is continuous on \( \overline{C} \). If we use \( T \) to denote \( T_l \), then the scattering coefficients from the left \( T \) and \( L \) are defined as in (4) or in (6) and (7). We need to modify (5), (8), (12), (14) and (22)-(24); the remaining displayed formulas in (2)-(25) remain unchanged. The potential \( V \) can be obtained by using the right Marchenko equation as in (49)-(53) together with the right scattering data \( \{ L, \{ \kappa \}, \{ c_{ij} \} \} \). The left Marchenko equation is still given by (44), but the formulas (45)-(48) need to be modified. The appropriate left scattering data consist of \( \xi \) for \( k \in \mathbb{R} \setminus [-c, c], \), \( T \) for \( k \in [-c, c] \), \( \kappa \) and \( c_{ij} \) for \( j = 1, \ldots, N \). We have

\[
\begin{align*}
\delta_l(\alpha) & := -\hat{R}(\alpha) - \frac{1}{2\pi} \int_0^c d\xi R(k(\xi)) e^{-\sqrt{k^2 - c^2} \alpha} \\
& - \sum_{j=1}^N c_{ij} e^{-\sqrt{k^2 + c^2} \alpha},
\end{align*}
\]

where

\[
\hat{R}(\alpha) := \frac{1}{2\pi} \int_{-\infty}^\infty d\xi R(k(\xi)) e^{\sqrt{k^2 - c^2} \alpha}, \quad (70)
\]

with \( k(\xi) \) denoting the map \( \xi \mapsto k \). The potential is obtained from the solution \( B_l(x, \alpha) \) of the Marchenko equation (44) as

\[
V(x) = c^2 - 2 \frac{dB_l(x, 0^+)}{dx}. \quad (71)
\]

The Jost solution from the left is constructed as

\[
f_l(k, x) = e^{ik\sqrt{k^2 - c^2} x} \left[ 1 + \int_0^\infty d\alpha B_l(x, \alpha) e^{i\sqrt{k^2 - c^2} \alpha} \right].
\]

We can give a characterisation of the scattering data \( \{ T_1, T_r, R, \kappa_1, \{ c_{ij} \}, \{ c_{ij} \} \} \) so that it corresponds to a unique potential \( V \) that is real valued and satisfies \( V \in L_1^2(R^-) \) and \( V - c^2 \in L_1^2(R^+) \). When the following conditions are satisfied, both the right and left Marchenko equations are uniquely solvable, and the right-hand sides of (47) and (71) are equal to each other, and \( V \) is obtained from (47) or (71).

(i) The right transmission coefficient \( T_r \) is related to the left transmission coefficient \( T := T_l \) as in (69), and the right reflection coefficient is related to the left scattering coefficients as in (30). The scattering coefficients \( T, R \) and \( L \) satisfy (10) and (11).

(ii) The reflection coefficients \( L \) and \( R \) are continuous on \( R \) and belong to \( L^2(R) \).

(iii) The left transmission coefficient \( T \) is meromorphic in \( C^* \) with simple poles at \( \kappa = i\kappa_1 \), it is continuous in \( C^* \setminus \{ i\kappa_1, \ldots, i\kappa_N \} \) and the residues of \( T(k) \) at \( k = i\kappa_j \) satisfy (19).

(iv) The scattering coefficients satisfy

\[
1 - |R(k)|^2 = 1 - |L(k)|^2 = \frac{\xi}{k} |T(k)|^2 \neq 0,
\]

\[
k \in \mathbb{R} \setminus [-c, c], \quad (72)
\]

\[
R(k) = -1, \quad L(k) = \frac{T(k)}{T^*(k)}, \quad k \in [-c, c]. \quad (73)
\]

(v) The reflection coefficients \( L \) and \( R \) are \( o(1/k) \) as \( k \to \pm \infty \). The left transmission coefficient \( T \)
is nonzero in $C^* \setminus \{0,i\kappa_1,\ldots,i\kappa_N\}$, and $T(k) = 1 + O(1/k)$ as $k \to \infty$ in $C^*$.

(vi) The scattering coefficients $T, R$ and $L$ are continuous at $k = 0$. As $k \to 0$ in $C^*$ we have (Aktosun, 1999)

$$T(k) = \begin{cases} \frac{2i k}{W_0} + o(1), & \text{generic case,} \\ \frac{2\gamma_0}{\gamma_0} + o(1), & \text{exceptional case,} \end{cases}$$

where $W_0$ and $\gamma_0$ are some real nonzero constants (they are actually the constants defined in (20) and (21), respectively), and as $k \to 0$ in $R$

$$R(k) = -1 + o(1), \quad L(k) = \begin{cases} 1 + o(1), & \text{generic case,} \\ 1 + o(1), & \text{exceptional case.} \end{cases}$$

(viii) $\hat{L}$ defined in (51) and $\hat{R}$ defined in (70) are absolutely continuous and belong to $L^2(R)$; moreover, $\hat{L}' \in L^1_{\text{loc}}(\infty, \alpha)$ and $\hat{R}' \in L^1_{\text{loc}}(\alpha, +\infty)$ for any fixed $\alpha \in R$.

### § 6. Phase Recovery

In X-ray and neutron reflectometry, we need to determine the scattering length density of an unknown layered material, and this is done by measuring the intensity of a probing beam reflected off that material as a function of the angle of incidence. Mathematically, this amounts to determining a portion of a steplike potential in the 1-D Schrödinger equation when we know the rest of the potential and some reflectivities (i.e., amplitudes of some reflection coefficients without their phases).

Consider (1) with a real-valued potential $V$ that can be written as $V = V_1 + V_2$, where $V_1$ is supported in $R^-$, $V_2$ is supported in $R^+$, $V_1 \in L^1_{\text{loc}}(R^-)$ and $V_2 - \sigma^2 \in L^1_{\text{loc}}(R^+)$ for some $\sigma \geq 0$. Let $R_1$ and $T_1$ denote the reflection coefficient from the right and the transmission coefficient for $V_1$, $L_2$ and $T_2$ the reflection and transmission coefficients from the left for $V_2$ and $L$ and $T$ the reflection and transmission coefficients from the left for $V$. For simplicity, we assume that neither $V_1$ nor $V_2$ has any bound states and refer the reader to Aktosun and Sacks (2000a) for the phase recovery problem with bound states.

Mathematically, we need to determine $V_2$ from the scattering data $\{R_1, L_2, |L|\}$. In other words, we are lacking the phases of $L_2$ and $L$ and only know their amplitudes. Because of (73), we have $|L_2(k)| = 1$ and $|L(k)| = 1$ for $k \in [-c, c]$. Thus, the data for $k \in [-c, c]$ are useful only to recover the value of $c$ and otherwise do not directly contribute to the phase recovery. As seen from §§5, $V_2$ is recovered when $L_2$ is obtained for $k \in R$ from the given data. This inverse problem can be solved as follows. From a generalisation of (40) to steplike potentials, we have (Aktosun and Sacks, 2000b)

$$\frac{1}{T(k)} = \frac{1 - R_1(k)L_2(k)}{T_1(k)T_2(k)}, \quad k \in R \setminus \{0\}. \quad (74)$$

Define

$$F(k) := 1 - R_1(k)L_2(k)^*, \quad k \in R. \quad (75)$$

From (74) we see that

$$F(k) = \frac{T_1(k)T_2(k)}{T(k)}. \quad (76)$$

Using (72) and (76) we get

$$|F|^2 = \frac{(1 - |R_1|^2)(1 - |L_2|^2)}{1 - |L|^2}, \quad k \in R \setminus [-c, c]. \quad (77)$$

On the other hand, from (75) it follows that

$$|F|^2 = 1 + |R_1|^2|L_2|^2 + 2\text{Re} F_1, \quad k \in R,$n

which, combined with (77) yields the real part of $F$ as

$$\text{Re} F = \frac{1}{2} \left[ 1 - |R_1|^2|L_2|^2 + \frac{(1 - |R_1|^2)(1 - |L_2|^2)}{1 - |L|^2} \right], \quad k \in R \setminus [-c, c]. \quad (78)$$

As seen from (77) and (78), our data uniquely determine both $|F|$ and $\text{Re} F$ at each $k \in R \setminus [-c, c]$. Moreover, because of (76), $F(k)$ has an analytic extension from $R$ to $C^*$ that is continuous in $C^*$ and free of zeros in $C^* \setminus \{0\}$. Thus, we can determine $F$ uniquely for $k \in C^*$. Then $L_2$ is recovered as $L_2 = (1 - F)/R_1$ for all $k \in R$, from which $V_2$ is uniquely constructed. Under certain restrictions, an exact quadrature is known (Aktosun and Sacks, 2000a) for the analytic continuation of $F$ or $L_2$ from $k \in R \setminus [-c, c]$ to $k \in [-c, c]$, and that quadrature is also easy to implement numerically.

Assume that $V_1$ is replaced by another potential $\tilde{V}_1$ whose reflection coefficient from the right is $\tilde{R}_1$. Let $\tilde{V} := \tilde{V}_1 + V_2$, and use $\tilde{L}$ to denote the reflection coefficient from the left for $\tilde{V}$. If we use $\{R_1, \tilde{R}_1, L_2, |L_2|, |L|\}$ as our scattering data, then

$$L_2 = \frac{R_1 - \tilde{R}_1 + |L_2|^2R_1\tilde{R}_1 [\tilde{R}_1 - R_1] + (1 - |L_2|^2)Y}{2i\text{Im} \{R_1\tilde{R}_1\}^*}, \quad k \in R \setminus [-c, c],$$

where we have defined

$$Y := \frac{\tilde{R}_1^* (1 - |\tilde{R}_1|^2)}{1 - |L|^2} \cdot \frac{R_1^* (1 - |R_1|^2)}{1 - |L|^2}.$$
\{R_1, |L_2|, |L_l|\} contain exactly the same information (Aktosun and Sacks, 2000b).

In the special case when \( V_2 \) is constant, we have \( L_2(k) = (k - \xi)/(k + \xi) \), where \( \xi \) is the quantity appearing in (69). Then, via (75), we have

\[
\text{Re} \{ R_1(k) \} = -\frac{1}{L_2(k)} \text{Re} \{ F(k) \}, \quad k \in \mathbb{R} \setminus [-c, c],
\]

and hence, because of the same (78), knowledge of \( \{ |R_1|, L_2, |L_l| \} \) at one \( k \) value with \( k > c \) uniquely and explicitly determines \( \text{Re} \{ R_1(k) \} \) at that \( k \) value. Moreover, \( R_1 \) has a unique analytic continuation from any interval on \( \mathbb{R} \) to \( \mathbb{C} \). Thus, knowledge of \( \text{Re} \{ R_1(k) \} \) on any such interval is sufficient to construct \( V_1 \). This is the basic idea behind the method of variation of surrounding media (Majkrzak and Berk, 1998), which is an experimentally feasible procedure in neutron reflectometry to determine the scattering length density of an unknown layered medium. The reader is referred to Majkrzak and Berk (1995, 1998), de Haan et al. (1995), Aktosun and Sacks (2000a,b) and the references therein for further studies on various physical and mathematical aspects of neutron reflectometry.

§ 7. Inverse Problems for Nonhomogeneous Media

Consider the generalised Schrödinger equation

\[
\frac{d^2 \psi(x, t)}{dx^2} + k^2 H(x)^2 \psi(x, t) = Q(x) \psi(x, t), \quad x \in \mathbb{R}, \tag{79}
\]

where \( Q \) is real valued and belongs to \( L^1(\mathbb{R}) \), \( H \) is real valued and strictly positive and \( H - 1 \in L^1(\mathbb{R}) \). Switching to the travel-time coordinate

\[
y = y(x) = \int_0^x dt \, H(t), \tag{80}
\]

via the Liouville transformation

\[
\phi(k, y) = \sqrt{H(x)} \psi(x, k), \tag{81}
\]

we can transform (79) into an equation of the form (1) that is given by

\[
\frac{d^2 \phi(k, y)}{dy^2} + k^2 \phi(k, y) = V(y) \phi(k, y), \quad y \in \mathbb{R}, \tag{82}
\]

with

\[
V(y(x)) := \frac{Q(x)}{H(x)^2} + \frac{H''(x)}{2H(x)^3} - \frac{3H'(x)^2}{4H(x)^4}. \tag{83}
\]

When \( V \in L^1(\mathbb{R}) \), we can recover \( Q(x) \) from any scattering data that imply knowledge of \( V(y) \) and the relation between \( x \) and \( y \).

Let \( \tau, \theta \) and \( \rho \) denote the transmission coefficient, the reflection coefficient from the left, and the reflection coefficient from the right, respectively, for (82), and let \( T, L \) and \( R \) denote the corresponding coefficients for (79). These two sets are related to each other by

\[
\tau = T(k) e^{ikA} \quad \theta = L(k) e^{2ikA} \quad \rho = R(k) e^{2ikA},
\]

where

\[
A_\pm := \pm \int_0^{\infty} dt \{ 1 - H(t) \} \quad A := A_- + A_+.
\]

Let \( f_1^{0(0)}(0, x) \) be the zero-energy Jost solution from the left corresponding to \( Q \). Note that \( Q \) uniquely determines \( f_1^{0(0)}(0, x) \) and vice versa. For example, given \( Q \) we can determine \( f_1^{0(0)}(0, x) \) by solving

\[
f_1^{0(0)}(0, x) = 1 + \int_{-\infty}^{\infty} dt \{ t - x \} Q(t) f_1^{0(0)}(0, t),
\]

and given \( f_1^{0(0)}(0, x) \) we can determine \( Q \) by using

\[
Q(x) = \frac{1}{f_1^{0(0)}(0, x)} \frac{d^2 f_1^{0(0)}(0, x)}{dx^2}. \tag{84}
\]

Note also that

\[
f_1^{0(0)}(0, x) = f_0(0, x), \quad x \in \mathbb{R}, \tag{85}
\]

where \( f_1(k, x) \) is the Jost solution from the left for (79).

Let \( Z_l(k, y) \) denote the Faddeev function from the left for (82); i.e., \( e^{iky} Z_l(k, y) \) is the Jost solution from the left for (82). Assuming that our scattering data consist of \( Q \), \( \rho \) and the bound-state energies and normalising constants for (82), we can recover \( V(y) \) by solving the Marchenko equation using this set of data as the input. Having found \( V(y) \), we also have \( Z_l(0, y) \) at hand because \( V(y) = Z_l^T(0, y)/Z_l(0, y) \) and

\[
Z_l(0, y) = 1 + o(1), \quad Z_l(0, y) = o(1), \quad y \to \pm \infty.
\]

The relationship between \( x \) and \( y \) can be obtained by solving the algebraic equation

\[
\int_0^y \frac{ds}{Z_l(0, s)} - \int_0^\infty \frac{dt}{f_0^{0(0)}(0, t)^2} = \tau \tag{86}
\]

which is obtained with the help of (81) at \( k = 0 \).

Once we have \( y = y(x) \), then \( H \) is obtained as \( H(x) = dy(x)/dx \).

On the other hand, suppose we are given \( H \) and are interested in the recovery of \( Q \). Using (80) we obtain \( y \) as a function of \( x \). Then from the data consisting of \( \rho \) and the bound-state energies and normalising constants, we obtain \( V(y) \). Having \( V(y) \) and \( y = y(x) \) at hand, we use (83) to recover \( Q(x) \).

The reader is referred to Aktosun et al. (1992a,b) and the references therein for the details and variations of the methods outlined above. Such methods also have generalisations when \( H \) and \( H' \) have jump discontinuities and \( H \) has distinct asymptotics as \( x \to \pm \infty \), which is discussed in § 8.
§ 8. Nonhomogeneous Media with Jump Discontinuities

Consider again the generalised Schrödinger equation (79) describing wave propagation in a nonhomogeneous, elastic medium. In order to consider abrupt changes in the material properties of the medium, \( H \) is now allowed to have jump discontinuities at a finite number of points. We also let \( H(x) \) have different asymptotics as \( x \to \pm \infty \). When \( Q \) is known, we are interested in reconstructing \( H \) from an appropriate set of scattering data, which will be specified.

The assumptions on \( H \) and \( Q \) are as follows:

(i) \( H \) is strictly positive and piecewise continuous with jump discontinuities at \( x_j \) for \( j = 1, \ldots, n \) such that \( x_1 < \cdots < x_n \).

(ii) \( H(x) \to H_\pm \) as \( x \to \pm \infty \), where \( H_\pm \) are positive constants.

(iii) \( H_+ - H_- \in L^1(\mathbb{R}^+) \).

(iv) \( H' \) is absolutely continuous on \((x_j, x_{j+1})\) and \( 2H'' - 3(H')^2 \in L^1_1(x_j, x_{j+1}) \) for \( j = 0, \ldots, n \), where \( x_0 := -\infty \) and \( x_{n+1} := +\infty \).

(v) \( Q \in L^1_1(\mathbb{R}) \).

(vi) There are no bound states for (79).

We can weaken (v) to just \( Q \in L^1_1(\mathbb{R}) \), which is sufficient (Aktosun et al., 1996a) for the reconstruction of \( H \). However, not all technical questions regarding the uniqueness of the inversion procedure (Aktosun et al., 1995, 1996a) have been answered when \( Q \) is only in \( L^1_1(\mathbb{R}) \). The inversion when there are bound states can be found in Aktosun et al. (1995). There are no bound states if, for example, \( Q(x) \geq 0 \). Because of (iv) a Liouville transformation can be used in each subinterval \((x_j, x_{j+1})\).

Various authors have studied inverse scattering problems for differential equations with discontinuous coefficients; for example, see Ware and Aki (1969), Krueger (1982), Sabatier (1988) and Grinberg (1991a, b). The work most directly related to the method presented here is that of Grinberg, who, in the special case \( Q = 0 \), analysed the recovery of \( H \) using the solution of a singular integral equation. When \( Q \neq 0 \) the analysis of the problem becomes more involved and there are essential differences in the results as compared to the case \( Q = 0 \).

There are some difficulties and new twists in this inverse problem that we do not encounter in the inverse problem for (1). There are also other interesting issues; for example, a question of practical importance is whether certain characteristic properties of \( H \) can be recovered more quickly, that is, without having to solve the inverse problem first. Quantities that fall into this category include the number of discontinuities, the ratios \( H(x_j^-)/H(x_j^+) \) and the integrals \( x_j^+ \int_0^{x_j} dz H(z) \) representing the time it takes for the wave to travel from the origin to the discontinuity \( x_j \). It turns out that such information can be extracted from the large-\( k \) asymptotics of the reflection and transmission coefficients (Grinberg, 1991a; Aktosun et al., 1996b).

The Jost solutions \( f_1 \) and \( f_2 \) of (79) satisfy

\[
\begin{align*}
f_1(k, x) &= \begin{cases} 
  e^{ikH_+} + o(1), & x \to +\infty, \\
  \frac{1}{T_1(k)} e^{ikH_+} + \frac{L(k)}{T_1(k)} e^{-ikH_-} + o(1), & x \to -\infty, 
\end{cases}

f_2(k, x) &= \begin{cases} 
  e^{-ikH_-} + o(1), & x \to -\infty, \\
  \frac{1}{T_2(k)} e^{-ikH_-} + \frac{R(k)}{T_2(k)} e^{ikH_+} + o(1), & x \to +\infty,
\end{cases}
\end{align*}
\]

where \( T_1 \) and \( T_2 \) are the transmission coefficients from the left and right, respectively, and \( L \) and \( R \) are the reflection coefficients from the left and right, respectively. Note that \( T_1(k) \) and \( T_2(k) \) are different unless \( H_+ = H_- \), and they are related by \( H_+ T_1(k) = H_- T_2(k) \).

As for (1) we need to distinguish between the generic case and the exceptional case; the definitions and notation are the same as in §1, and the division into a generic case and an exceptional case is solely governed by \( Q \) because \( H \) does not affect the solutions of (79) at \( k = 0 \). In particular, if \( Q \equiv 0 \) then the exceptional case occurs, and if \( Q(x) \geq 0 \) but \( Q \equiv 0 \) then the generic case occurs.

Using the travel-time coordinate (80), we can still transform (79) into (82); even though the potential \( V(y) \) given in (83) cannot be defined at \( y_j := y(x_j) \), we have \( V \in L^1_1(y_j, y_{j+1}) \) for \( j = 0, \ldots, n \). The function \( \phi(k, y) \) defined in (81) and \( \phi'(k, y) \) are not continuous at \( y_j \), but they satisfy certain matching conditions (Aktosun et al., 1995) involving the jumps in \( H \) and in \( H'/H \).

Define the Faddeev functions \( Z_1(k, y) \) and \( Z_2(k, y) \) associated with (79) as

\[
\begin{align*}
Z_1(k, y) &:= \sqrt{\frac{H(x)}{H_+}} e^{-iky - ikA_y} f_1(k, x), \\
Z_2(k, y) &:= \sqrt{\frac{H(x)}{H_-}} e^{-iky - ikA_y} f_2(k, x),
\end{align*}
\]

where

\[
A_y := \pm \int_0^{\pm\infty} dz [H(x) - H(z)], \quad A := A_- + A_+.
\]

For each \( y \in \mathbb{R} \setminus \{y_1, \ldots, y_n\} \), the Faddeev functions \( Z_1(k, y) \) and \( Z_2(k, y) \) are continuous on \( C^1 \), are analytic on \( C^1_+ \), behave like \( 1 + O(1/k) \) as \( k \to \infty \) in \( C^2 \) and satisfy the Riemann–Hilbert problem

\[
\begin{bmatrix}
Z_1(-k, y) \\
Z_2(-k, y)
\end{bmatrix} = \begin{bmatrix}
\tau(k) & -\overline{\phi(k)} e^{2i\pi y} \\
-\overline{\phi(k)} e^{-2i\pi y} & \tau(k)
\end{bmatrix} \begin{bmatrix}
Z_1(k, y) \\
Z_2(k, y)
\end{bmatrix},
\]

\( k \in \mathbb{R}, \) (86)
where the reduced scattering coefficients are given by
\[ \tau(k) := \sqrt{\frac{H_\infty}{H_*}} T_1(k) e^{ikA}, \quad \phi(k) := L(k) e^{i2kA}. \] (87)

Define
\[ X(k, y) := \frac{i \sqrt{H_*}}{k |H(x)| f_0(x)} \left[ Z_1(-k, y) - Z_1(0, y) \right]. \] (88)

The notation is justified, since it turns out that the right-hand side of (88) does not depend explicitly on \( x \), so the left-hand side depends on \( x \) only through \( y \). The Riemann–Hilbert problem (86) can be transformed into the singular integral equation
\[ X(k, y) = X_0(k, y) + (\partial_y X)(k, y), \] (89)
where
\[ X_0(k, y) := \frac{1}{2 \pi i} \int_{-\infty}^{\infty} \frac{ds}{s + k + i0^+} p(s) e^{i\pi y} - p(0), \] (90)
\[ (\partial_y X)(k, y) := \frac{1}{2 \pi i} \int_{-\infty}^{\infty} \frac{ds}{s + k + i0^+} p(-s) e^{i\pi y} X(s, y). \] (91)

The integral equation (89) is uniquely solvable in the Hardy spaces \( H^p_\infty(R) \) with \( 1 < p < +\infty \) when \( Q \in L^1_\infty(R) \). Under the weaker assumption \( Q \in L^1_\infty(R) \), the unique solvability holds for \( p < (1 - \alpha)^{-1} \).

Note that (89) uses \( p(k) \) as the input. In order to use \( R(k) \) as the input in (89), we can exploit (87), where \( p(k) \) and \( R(k) \) are related by the phase factor involving \( A_* \), which can be handled by a shift in \( y \) in the solution of (89). If \( X(k, y) \) denotes the solution of (89) obtained by replacing \( p(k) \) by \( R(k) \) in (90) and (91), then we get
\[ X(k, y - A_*) = X(k, y). \] (92)

Using (86), (88), and (92) we obtain
\[ \gamma + A_* + \tilde{X}(0, \gamma + A_*) = H_* G_1(x), \] (93)
where (Aktosun et al., 1996b)
\[ G_1(x) := \int_0^\infty \frac{dz}{z^2 + \tilde{f}(0, z)^2} + \int_0^\infty \frac{dz}{\tilde{f}(0, z)^2}. \]

When \( Q \in L^1_\infty(R) \), we have \( 1 - f_1(0, \cdot)^2 \in L^1(R^+) \); because of (84) and (85), \( G_1(x) \) is determined by \( Q(x) \). We remark that (93) is an implicit equation for \( y(x) \). The constant \( A_* \) is determined by using the condition \( y(0) = 0 \) in (80), which results in
\[ A_* + \tilde{X}(0, A_*) = H_* G_1(0). \] (94)

It can be shown that this equation determines \( A_* \) uniquely (Aktosun et al., 1996b).

The recovery of \( H \) is accomplished as follows. First, we solve (89) with \( p(k) \) replaced by \( R(k) \) and obtain \( \tilde{X}(k, y) \). Next, (94) is solved for \( A_* \) and then
\[ y(x) \) is obtained by solving (93). As a final step, we obtain \( H(x) = y'(x) \) by differentiating \( y(x) \).

We can analyze (93) in more detail (Aktosun et al., 1996b). In the exceptional case, the constant \( H_* \) is a free parameter and must be specified for a unique recovery of \( H \). In the generic case, the situation is a bit different. The limits \( w_0 := \lim_{x \to \infty} [z + \tilde{X}(0, z)] \) and \( G_1(-\infty) := \lim_{x \to -\infty} G_1(x) \) exist and are finite, so that, by (93), we have \( H_* = w_0 / G_1(-\infty) \) provided \( G_1(-\infty) \neq 0 \); in other words, \( H_* \) is not a free parameter, and it is fixed by the data \( \{R, Q\} \). However, if \( G_1(-\infty) = 0 \), we must also have \( w_0 = 0 \) in order for (93) to be solvable for \( y \) as a function of \( x \). We can show that if \( G_1(-\infty) = 0 \), then \( H_* \) is a free parameter as in the exceptional case. From these facts it follows that the proper scattering data to recover \( H \) are given by \( \{R, H_*, Q\} \) in the exceptional case, by \( \{R, H_*, Q\} \) in the generic case with \( G_1(-\infty) = 0 \), and by \( \{R, Q\} \) in the generic case with \( G_1(-\infty) \neq 0 \).

In summary, under the assumptions (i)–(vi), the solution of the inverse scattering problem is unique when the scattering data are chosen as indicated above. Moreover, both in the generic case with \( G_1(-\infty) = 0 \) and in the exceptional case, the constant \( H_* \) is a free parameter in the sense that for any choice of \( H_* > 0 \), the function \( H \) resulting from the solution of (93) corresponds to the same reflection coefficient \( R \).

For the recovery of \( H \) with jump discontinuities by using a Marchenko method, the reader is referred to Aktosun et al. (1996a).

### § 9. Other Equations

Consider the impedance-potential equation (Sabatier, 1988; Chadan and Sabatier, 1989)
\[ \frac{1}{p(x)^2} \frac{d}{dx} \left( p(x)^2 \frac{d\psi(x)}{dx} \right) + k^2 \psi(x) = Q(x) \psi(x), \] (95)
\[ x \in \mathbb{R}. \]

where \( Q \) is real valued, \( Q \in L^1_\infty(R) \), \( p \) is strictly positive, \( p^n \in L^1_\infty(R) \) and \( p - 1 \in L^1_\infty(R) \). Letting \( \phi(x) := p(x)^{1/2} \psi(x) \), we can transform (95) into an equation of the form (1) that is given by
\[ \frac{d^2\phi(x)}{dx^2} + k^2 \phi(x) = \left[ Q(x) + \frac{p^n(x)}{p(x)} \right] \phi(x), \] \[ x \in \mathbb{R}. \]

Alternatively, using
\[ y = y(x) := \int_0^x \frac{dt}{p(t)^{1/2}}, \quad \psi(k, y(x)) = \psi(x, k), \]
we can transform (95) into an equation similar to (79) that is given by
\[ \frac{d^2\phi(x, y)}{dy^2} + k^2 n(y)^2 \phi(x, y) = W(y) \psi(k, y), \] \[ y \in \mathbb{R}, \]
where
\[ n(y) := \int_0^y \frac{dt}{p(t)^{1/2}}. \]
where

\[ n(y(x)) := p(x)^2, \quad W(y(x)) := Q(x)p(x)^4. \]

The reader is referred to Chadan and Sabatier (1989) for the analysis of (95) when \( p(x) \) and \( p'(x) \) have jump discontinuities.

For the inverse scattering problems related to the matrix Schrödinger equation, the reader is referred to Olmedilla (1985), Chadan and Sabatier (1989) and the references therein. Under certain restrictions, the \( n \times n \) matrix potential is recovered via the Marchenko method from an appropriate set of scattering data containing an \( n \times n \) matrix reflection coefficient and the bound-state information.

Consider the inverse scattering problem of recovery of \( P(x) \) and \( Q(x) \) in

\[ \psi''(k,x) + k^2 \psi(k,x) = [ikP(x) + Q(x)]\psi(k,x), \quad x \in \mathbb{R}. \quad (96) \]

When \( P(x) \) is purely imaginary and \( Q(x) \) is real the reader is referred to Jaulent and Jean (1976a,b) and Sattinger and Szmigielski (1995). A Marchenko inversion method can be formulated utilising the scattering data from (96) as well as the data from the equation obtained when \( P(x) \) is changed to \(-P(x)\) in (96). If \( P \) is real valued, then (96) is non-self-adjoint, and the inverse problem (Jaulent, 1976; Aktosun et al., 1998a) becomes challenging due to the nonunitarity of the scattering matrix, possible singularities of the transmission coefficient at real \( k \) values and some complications related to bound states.

For the inverse problems related to first-order systems containing a matrix potential, the reader is referred to Beals and Coifman (1987), Chadan and Sabatier (1989) and Aktosun et al. (2000) as starting references for further reading. Such inverse problems are usually formulated as a matrix Riemann–Hilbert problem, from whose solution the matrix potential is constructed.

The connection between the inverse scattering for (1) and the initial-value problem for the Korteweg-de Vries equation (KdV) is treated in Chapter 6.2.1. Here we briefly mention the recent work by Sabatier (1999) on the recovery of the solution \( u(x,t) \) of the KdV equation for all \( x, t \in \mathbb{R} \) when \( u(0, t), u_x(0, t) \) and \( u_{xx}(0, t) \) are known for all \( t \in \mathbb{R} \), and the study on "elbow scattering" (Sabatier, 2000a) related to the KdV and linearised KdV equations, where the scattering and inverse scattering problems are analysed on two perpendicular half-lines meeting at a point on the \( xt \) plane.

For various other inverse problems on the line, we refer the reader to Chapter 17 of Chadan and Sabatier (1989), the recent review paper by Sabatier (2000b) and the references therein.


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Klaus, M. 1988, Low-energy behaviour of the scattering matrix for the Schrödinger equation on the line. *Inverse Problems* 4, 505–512.


